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Parallel spinors on pseudo-Riemannian *spin*^{c} manifolds☆

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Abstract

We describe, by their holonomy groups, all simply connected irreducible non-locally symmetric pseudo-Riemannian *spin*^{c} manifolds which admit parallel spinors. So we generalize the Riemannian *spin*^{c} case [A. Moroianu, Parallel and killing spinors on *spin*^{c} manifolds, Commun. Math. Phys. 187 (1997) 417–427] and the pseudo-Riemannian *spin* one [1].

MSC: 53C50; 53C27

Keywords: Holonomy groups; Pseudo-Riemannian *spin*^{c} manifolds; Parallel spinors

1. Introduction

In [13], Moroianu described all simply connected Riemannian *spin*^{c} manifolds admitting parallel spinors. Precisely, he showed the following result:

Theorem 1. *A simply connected Riemannian spin*^{c} *manifolds* (M, g) *admits a parallel spinor if and only if it is isometric to the Riemannian product* $(M_1, g_1) \times (M_2, g_2)$ *of a*

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Table 1

Holonomy group	N
$SU(p', q') \subset SO(2p', 2q')$	2
$Sp(p', q') \subset SO(4p', 4q')$	$p' + q' + 1$
$G_2 \subset SO(7)$	1
$G'_{2(2)} \subset SO(4, 3)$	1
$G_2^{\mathbb{C}} \subset SO(7, 7)$	2
$Spin(7) \subset SO(8)$	1
$Spin(4, 3) \subset SO(4, 4)$	1
$Spin(7, \mathbb{C}) \subset SO(8, 8)$	1

complete simply connected Kähler manifold (M_1, g_1) *and a complete simply connected spin manifold* (M_2, g_2) *admitting a parallel spinor. The spin*^{c} *structure of* (M, g) *is then the product of the canonical spin*^{c} *structure of* (M_1, g_1) *and the spin structure of* (M_2, g_2).

In [1], Baum and Kath characterized, by their holonomy group, all simply connected irreducible non-locally symmetric pseudo-Riemannian *spin* manifolds admitting parallel spinors. Precisely, they proved the following result:

Theorem 2. *Let* (M, g) *be a simply connected irreducible non-locally symmetric pseudo-Riemannian spin manifold of dimension* $n = p + q$ *and signature* (p, q). *We denote by* N *the dimension of the space of parallel spinors on* M. *Then* (M, g) *admits a parallel spinors if and only if the holonomy group* H *of* M *is* (*up to conjugacy in* $O(p, q)$) *one in the* Table 1:

Our aim is to generalize this result for the simply connected irreducible non-locally symmetric pseudo-Riemannian *spin*^{c} manifolds. More precisely, we show that:

Theorem 3. *Let* (M, g) *be a connected simply connected irreducible non-locally symmetric spin*^{c} *pseudo-Riemannian manifold of dimension* $n = p + q$ *and signature* (p, q). *Then the following conditions are equivalent*

- (i) (M, g) *admits a parallel spinor*,
- (ii) *either* (M, g) *is a spin manifold which admit a parallel spinor, or* (M, g) *is a Kähler not Ricci-flat manifold*,
- (iii) *the holonomy group* H *of* (M, g) *is* (*up to conjugacy in* $O(p, q)$) one in Table 1 or $H = U(p', q')$, $p = 2p'$ and $q = 2q'$.

For $H = U(p', q')$ *the dimension of the space of parallel spinors on* M *is 1.*

This theorem is a contribution to the resolution of the following problem: (**P**) What are the possible holonomy groups of simply connected pseudo-Riemannian *spin*^{c} manifolds which admit parallel spinors? Some partial answers to this problem have been given by Wang for the Riemannian *spin* case [15], by Baum and Kath for the irreducible pseudo-Riemannian *spin* one [1], by Leistner for the Lorentzian *spin* one [10,12], by Moroianu for the Riemannian *spin*^{c} one (Theorem 1), and by author for the totally reducible pseudo-

Riemannian *spin* one and the Lorentzian $spin^c$ one [7,8]. The problem remains open even though big progress have been made because the classification of the possible holonomy groups of pseudo-Riemannian manifolds is not yet made with the exception of the irreducible case made by Berger [3,4] and the case of Lorentzian manifolds made by Bérard Bergery, the author, Leistner and Galaev [2,11,12,6]. By De Rham-Wu's splitting theorem, the problem (**P**) can be reduced to the case of the indecomposable pseudo-Riemannian manifolds (its holonomy representation does not leave invariant any non-degenerate proper subspace). But the general classification remains extremely difficult, because some indecomposable but non irreducible manifolds exist, i.e. its holonomy representation leaves invariant a degenerate proper subspace but its does not leave invariant any non-degenerate proper subspace. In this article we deal with studying the irreducible case that is a particular case of the indecomposable one.

In paragraph 2 of this paper we define the group $Spin^c(p,q)$ and its spin representation. We also define the $spin^c$-structure on pseudo-Riemannian manifolds and its associated spinor bundle. In paragraph 2 we give an algebraic characterization of the pseudo-Riemannian $spin^c$ manifolds which admit parallel spinors and we prove Theorem 3.

2. Spinor representations and $spin^c$-bundles

2.1. $Spin^c(p,q)$ groups

Let $\langle\cdot,\cdot\rangle_{p,q}$ be the ordinary scalar product of signature (p,q) on $\mathbb{R}^m (m=p+q)$. Let $Cl_{p,q}$ be the Clifford algebra of $\mathbb{R}^{p,q} := (\mathbb{R}^m, \langle\cdot,\cdot\rangle_{p,q})$ and $\mathbb{C}l_{p,q}$ its complexification. We denote by $\cdot$ the Clifford multiplication of $\mathbb{C}l_{p,q}$. $\mathbb{C}l_{p,q}$ contains the groups:

$$\mathbb{S}^1 := \{z \in \mathbb{C}; \| z \| = 1\}$$

and

$$Spin(p,q) := \{X_1, \ldots, X_{2k}; \langle X_i, X_i\rangle_{p,q} = \pm 1; k \geq 0\}.$$

Since $\mathbb{S}^1 \cap Spin(p,q) = \{-1, 1\}$, we define the group $Spin^c(p,q)$ by:

$$Spin^c(p,q) := Spin(p,q) \cdot \mathbb{S}^1 = Spin(p,q) \times_{\mathbb{Z}_2} \mathbb{S}^1.$$

Consequently, the elements of $Spin^c(p,q)$ are the classes $[g,z]$ of pairs $(g,z) \in Spin(p,q) \times \mathbb{S}^1$, under the equivalence relation $(g,z) \sim (-g,-z)$. The following sequences are exact (see [9]):

$$1 \to \mathbb{Z}_2 \to Spin(p,q) \xrightarrow{\lambda} SO(p,q) \to 1$$

$$1 \to \mathbb{Z}_2 \to Spin^c(p,q) \xrightarrow{\xi} SO(p,q) \times \mathbb{S}^1 \to 1,$$

where $\lambda(g)(x) = g \cdot x \cdot g^{-1}$ for $x \in \mathbb{R}^m$ and $\xi([g, z]) = (\lambda(g), z^2)$. Let $(e_i)_{1\le i\le m}$ be an orthonormal basis of $\mathbb{R}^{p,q}$ ($\langle e_i, e_j\rangle = \varepsilon_i\delta_{ij}$, $\varepsilon_i = -1$ for $1 \le i \le p$ and $\varepsilon_i = +1$ for $1 + p \le i \le m$). The Lie algebras of $Spin(p, q)$ and $Spin^c(p, q)$ are respectively:

$$spin(p, q) := \{e_i \cdot e_j; 1 \le i < j \le m\}$$

and

$$spin^c(p, q) := spin(p, q) \oplus \mathrm{i}\mathbb{R}.$$

The derivative of ξ is a Lie algebra isomorphism and it is given by:

$$\xi_*(e_i \cdot e_j, \mathrm{i}t) = (\lambda_*(e_i \cdot e_j), \mathrm{i}t) = (2E_{ij}, 2\mathrm{i}t),$$

where $E_{ij} = -\varepsilon_j D_{ij} + \varepsilon_i D_{ji}$ and D_{ij} is the standard basis of $gl(m, \mathbb{R})$ with the (i, j)-component equal 1 and all other zero.

2.2. *Spin^c representations*

Let $U = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, and $\mathbb{C}(2^n)$ the complex algebra consisting of $2^n \times 2^n$-matrices. It is well know that the Clifford algebra $\mathbb{C}l_{p,q}$ is isomorphic to $\mathbb{C}(2^n)$ if $m = p + q$ is even and to $\mathbb{C}(2^n) \oplus \mathbb{C}(2^n)$ if m is odd. Some natural isomorphisms are defined like follows (see [1]). In case $m = 2n$ is even, we define $\Phi_{p,q} : \mathbb{C}l_{p,q} \to \mathbb{C}(2^n)$ by:

$$\Phi_{p,q}(e_{2j-1}) = \tau_{2j-1} E \otimes \cdots \otimes E \otimes U \otimes T \otimes \cdots \otimes T$$

$$\Phi_{p,q}(e_{2j}) = \tau_{2j} E \otimes \cdots \otimes E \otimes V \otimes \underbrace{T \otimes \cdots \otimes T}_{(j-1)-\text{times}}, \tag{1}$$

where $\tau_j = \mathrm{i}$ if $\varepsilon_j = -1$ and $\tau_j = 1$ if $\varepsilon_j = 1$. In case $m = 2n + 1$ is odd, $\Phi_{p,q} : \mathbb{C}l_{p,q} \to \mathbb{C}(2^n) \oplus \mathbb{C}(2^n)$ is defined by:

$$\Phi_{p,q}(e_k) = (\Phi_{p,q-1}(e_k), \Phi_{p,q-1}(e_k)), k = 1, \ldots, m - 1;$$
$$\Phi_{p,q}(e_m) = (\mathrm{i}T \otimes \cdots \otimes T, -\mathrm{i}T \otimes \cdots \otimes T). \tag{2}$$

This yields representations of the spin group $Spin(p, q)$ in case m even by restriction and in case m odd by restriction and projection onto the first component. The module space of $Spin(p, q)$-representation is $\Delta_{p,q} = \mathbb{C}^{2^n}$. The Clifford multiplication is defined by:

$$\text{if } m \text{ is even} \quad X \cdot u := \Phi_{p,q}(X)(u),$$
$$\text{if } m \text{ is odd} \quad X \cdot u := pr_1 \Phi_{p,q}(X)(u), \tag{3}$$

for $X \in \mathbb{C}^m$ and $u \in \Delta_{p,q}$, where pr_1 is the projection onto the first component. A usual basis of $\Delta_{p,q}$ is the following: $u(\nu_n, \ldots, \nu_1) := u(\nu_n) \otimes \cdots \otimes u(\nu_1); \nu_j = \pm 1$, where

$$u(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u(-1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2.$$

The spin representation of the group $Spin(p, q)$ extends to a $Spin^c(p, q)$-representation by:

$$\begin{aligned} \Phi_{p,q}([g, z])(v) &= z\Phi_{p,q}(g)(v) := zg \cdot v, \quad \text{if } m \text{ is even}, \\ \Phi_{p,q}([g, z])(v) &= z\, pr_1 \Phi_{p,q}(g)(v) := zg \cdot v, \quad \text{if } m \text{ is odd}, \end{aligned} \tag{4}$$

for $v \in \Delta_{p,q}$ and $[g, z] \in Spin^c(p, q)$. Therefore $\Delta_{p,q}$ becomes the module space of $Spin^c(p, q)$-representation (see [5]).

There exists a hermitian inner product $\langle \cdot, \cdot \rangle_\Delta$ on the spinor module $\Delta_{p,q}$ defined by:

$$\langle v, w \rangle_\Delta := \mathrm{i}^{p(p-1)/2}(e_1, \ldots, e_p v, w); \quad \text{for} \quad v, w \in \Delta_{p,q},$$

where $(z, z') = \sum_{i=1}^{2^n} z_i \cdot \overline{z'_i}$ is the standard hermitian product on $\mathbb{C}^{2^n}$. $\langle \cdot, \cdot \rangle_\Delta$ satisfies the following properties:

$$\langle X \cdot v, w \rangle_\Delta = (-1)^{p+1} \langle v, X \cdot w \rangle_\Delta, \tag{5}$$

for $X \in \mathbb{C}^m$.

2.3. Spinor bundles

Let (M, g) be a connected pseudo- Riemannian oriented manifold of signature (p, q). And let $P_{SO(p,q)}$ denote the bundle of positively oriented frames on M.

Definition 1. A structure *spin* on (M, g) is a λ-reduction $P_{Spin(p,q)}$ of $P_{SO(p,q)}$. A structure *spin*^{c} on (M, g) is a $\mathbb{S}^1$-principal bundle $P_{\mathbb{S}^1}$ over M and a ξ-reduction $(P_{Spin^c(p,q)}, \Lambda)$ of the product $(SO(p, q) \times \mathbb{S}^1)$-principal bundle $P_{SO(p,q)} \tilde{\times} P_{\mathbb{S}^1}$, i.e. $\Lambda : P_{Spin^c(p,q)} \to (P_{SO(p,q)} \tilde{\times} P_{\mathbb{S}^1})$ is a two-fold covering verifying:

(i) $P_{Spin^c(p,q)}$ is a $Spin^c(p, q)$-principal bundle over M,
(ii) $\forall u \in P_{Spin^c(p,q)}, \forall a \in Spin(p, q)$,

$$\Lambda(ua) = \Lambda(a)\xi(a).$$

We note that if (M, g) is a space- and time-oriented manifold its carries a *spin*^{c}-structure if and only if the second Stiefl–Whitney class of M, $w_2(M) \in H^2(M, \mathbb{Z})$ is the $\mathbb{Z}_2$ reduction of an integral class $z \in H^2(M, \mathbb{Z}_2)$ [9,5].

Example 1. Every pseudo-Riemannian *spin* manifold is canonically a *spin*^{c} manifold. The *spin*^{c}- manifold is obtained as:

$$P_{Spin^c(p,q)} = P_{Spin(p,q)} \times_{\mathbb{Z}_2} \mathbb{S}^1,$$

where $P_{Spin(p,q)}$ is the Spin-bundle and $\mathbb{Z}_2$ acts diagonally by $(-1, -1)$.

Example 2. Any irreducible pseudo-Riemannian not Ricci-flat Kähler manifold is canonically a *spin*^{c} manifold.

Indeed the holonomy group H of (M, g) is $U(p', q')$, where $(p, q) = (2p', 2q')$ is the signature of (M, g). Then $P_{SO(p,q)}$ is reduced to the holonomy $U(p', q')$-principal bundle $P_{U(p',q')}$. Moreover, there exists an $\langle\cdot,\cdot\rangle_{p,q}$-orthogonal complex structure J on $\mathbb{R}^{p,q}$ witch commute with the elements of $U(p', q')$. Then there exist elements $(e_k)_{k=1,\ldots,p'+q'}$ such that $(e_k, Je_k)_{k=1,\ldots,p'+q'}$ is an orthonormal basis of $\mathbb{R}^{p,q}$. Hence we can imbed $U(p', q')$ in $SO(p, q)$ by

$$\mathrm{i} : U(p', q') \hookrightarrow SO(p, q)$$

$$A + \mathrm{i}B = ((a_{kl})_{1\le k,l\le m} + \mathrm{i}(b_{kl})_{1\le k,l\le m}) \to \left(\begin{pmatrix} a_{kl} & b_{kl} \\ -b_{kl} & a_{kl} \end{pmatrix}\right)_{1\le k,l\le m}.$$

We consider the homomorphism

$$\alpha : U(p', q') \hookrightarrow SO(p, q) \times \mathbb{S}^1 \qquad C \to (i(C), \det(C))$$

The eigen values of every element $C \in U(p', q')$ is in $\mathbb{S}^1$ and

$$\cos 2\theta + \varepsilon_k \sin 2\theta e_k \cdot Je_k = \varepsilon_k(\cos\theta e_k + \sin\theta Je_k)(-\cos\theta e_k + \sin\theta Je_k),$$

where $\varepsilon_k = \langle e_k, e_k\rangle_{p,q}$. Then the following homomorphism is well defined:

$$\tilde{\alpha} : U(p', q') \hookrightarrow Spin^c(p, q) \qquad C \to \prod_{k=1}^{m} \left(\cos\frac{\theta_k}{2} + \varepsilon_k \sin\frac{\theta_k}{2} e_k \cdot Je_k\right) \mathrm{e}^{\mathrm{i}/2\sum\theta_k},$$

where $\mathrm{e}^{\mathrm{i}\theta_k}$, $k = 1, \ldots, m$, are the eigen values of C. And it is easy to verifies that the following diagram commutes

$$\begin{array}{ccc} & & Spin^c c(p,q) \\ & \overset{\tilde{\alpha}}{\nearrow} & \downarrow \xi \\ U(p',q') & \overset{\alpha}{\longrightarrow} & SO(p,q) \times \mathbb{S}^1. \end{array} \tag{6}$$

Consequently,

$$P_{Spin^c(p,q)} = P_{U(p',q')} \times_{\tilde{\alpha}} Spin^c(p, q).$$

Now, let us denote by $S := P_{Spin^c(p,q)} \times_{\Phi_{p,q}} \Delta_{p,q}$ the spinor bundle associated to the *spin*^{c}-structure $P_{Spin(p,q)}$. The Clifford multiplication given by (3) defines a Clifford multiplication on S:

$$TM \otimes S = (P_{Spin(p,q)} \times_{\Phi_{p,q}} \mathbb{R}^m) \otimes (P_{Spin(p,q)} \times_{\Phi_{p,q}} \Delta^{\pm}_{p,q}) \to S$$
$$(X \otimes \psi) = [q, x] \otimes [q, v] \to [q, xv] =: X \cdot \psi.$$

The scalar product $\langle\cdot,\cdot\rangle_\Delta$ is $Spin_0^c(p,q)$-invariant. Then if we suppose that (M, g) is a space- and time-oriented manifold $\langle\cdot,\cdot\rangle_\Delta$ defines a scalar product on S by:

$$\langle\psi, \psi_1\rangle_\Delta = \langle v, v_1\rangle_\Delta, \quad \text{for} \quad \psi = [q, v] \quad \text{and} \quad \psi_1 = [q, v_1] \in \Gamma(S).$$

According to (5), it is then easy to verify that

$$\langle X \cdot \psi, \psi_1\rangle_\Delta = (-1)^{p+1}\langle\psi, X \cdot \psi_1\rangle_\Delta, \tag{7}$$

for $X \in \Gamma(M)$ and $\psi, \psi_1 \in \Gamma(S)$. Now, as in the Riemannian case (see [5]), if (M, g) is a pseudo- Riemannian *spin*^{c} manifold, every connection form $A : TP_{\mathbb{S}^1} \to \mathrm{i}\mathbb{R}$ on the $\mathbb{S}^1$-bundle $P_{\mathbb{S}^1}$ defines (together with the Levi–Civita D of (M, g)) a covariant derivative ∇^A on the spinor bundle S, called the spinor derivative associated to $(M, g, S, P_{\mathbb{S}^1}, A)$.

Henceforth, a pseudo- Riemannian *spin*^{c} manifold will be a set $(M, g, S, P_{\mathbb{S}^1}, A)$, where (M, g) is an oriented connected pseudo-Riemannian manifold, S is a *spin*^{c} structure, $P_{\mathbb{S}^1}$ is the $\mathbb{S}^1$-principal bundle over M and A is a connection form on $P_{\mathbb{S}^1}$. Using (7) and by the same proof in the Riemannian case (see [5]), we conclude that

Proposition 1. $\forall X, Y \in \Gamma(M)$ *and* $\forall\psi, \psi_1 \in \Gamma(S)$,

$$\nabla^A_Y(X \cdot \psi) = X \cdot \nabla^A_Y(\psi) + D_Y X \cdot \psi. \tag{8}$$

And if we suppose that (M, g) *is space- and time-oriented manifold*,

$$X\langle\psi, \psi_1\rangle_\Delta = \langle\nabla^A_X\psi, \psi_1\rangle_\Delta + \langle\psi, \nabla^A_X\psi_1\rangle_\Delta. \tag{9}$$

Let us denoted by $F_A := \mathrm{i}w$ the curvature form of A, seen as an imaginary-valued two-form on M, by R and Ric, respectively, the curvature and the Ricci tensors of (M, g) and by R^A the curvature tensor of ∇^A. Like in the Riemannian case (see [5]), if we put $\tilde{\omega}(X) := X \lrcorner \omega$ we have

Proposition 2. *For* $q = (e_1, \ldots, e_m)$ *a local section of* $P_{Spin(p,q)}$, $\forall X, Y \in \Gamma(M)$ *and* $\forall\psi \in \Gamma(S)$,

$$R^A(X, Y)\psi = \frac{1}{2}\sum_{1\le i<j\le m} \varepsilon_i\varepsilon_j\, g(R(X, Y)e_i, e_j)e_i \cdot e_j \cdot \psi + \mathrm{i}\frac{1}{2}\omega(X, Y) \cdot \psi, \tag{10}$$

and

$$\sum_{1\le i\le m} \varepsilon_i e_i \cdot R^A(X, e_i)\psi = -\frac{1}{2}Ric(X) \cdot \psi + \mathrm{i}\frac{1}{2}\tilde{\omega}(X) \cdot \psi. \tag{11}$$

Remark 1. According to Example 1, if (M, g) is *spin* then it is *spin*^{c}. Moreover, the auxiliary bundle $P_{\mathbb{S}^1}$ is trivial and then there exists a global section $\sigma : M \to P_{\mathbb{S}^1}$. We choose the connection defined by A to be flat, and we denote ∇^A by ∇. Conversely, if the auxiliary bundle $P_{\mathbb{S}^1}$ of a *spin*^{c}-structure is trivial, it is canonically identified with a *spin*-structure. Moreover, if the connection A is flat, by this identification, ∇^A corresponds to the covariant derivative on the spinor bundle.

3. Parallel spinors

3.1. Algebraic characterization

It is well know that there exists a bijection between the space $\mathcal{PS}$ of all parallel spinors on (M, g) and the space

$$V_{\tilde{H}} = \{v \in \Delta_{p,q}; \tilde{H} \cdot v = v\}$$

of all fixed spinors of $\Delta_{p,q}$ with respect to the holonomy group $\tilde{H}$ of the connection ∇^A [1]. If (M, g) is supposed to be simply connected, then $\mathcal{PS}$ is in bijection with:

$$V_{\mathcal{H}} = \{v \in \Delta_{p,q}; \tilde{\mathcal{H}} \cdot v = 0\},$$

where $\tilde{\mathcal{H}}$ is the Lie algebra of $\tilde{H}$. Moreover, provided with the connection defined by the Levi–Civita connection D and the connection form A the holonomy group of $P_{SO}(p, q)\tilde{\times} P_{\mathbb{S}^1}$ is $\xi(\tilde{H}) \subset H \times H_A$, where H is the holonomy group of (M, g) and H_A the one of A (see Chapter II, [14]). $H_A = \{1\}$ if A is flat and $H_A = \mathbb{S}^1$ otherwise. With the notations introduced in Section 2.1, for $(B, it) \in \xi_*(\tilde{\mathcal{H}})$, we have

$$\xi_*^{-1}(B, \mathrm{i}t) = \left(\lambda_*^{-1}(B), \frac{1}{2}\mathrm{i}t\right).$$

Now if we differentiate the relation (4) at [1,1], we get:

$$\phi_{p,q}(C, \mathrm{i}t)(v) = \mathrm{i}tv + \phi_{p,q}(C)(v),$$

for $(C, it) \in spin^c(p, q)$ and $v \in \Delta_{p,q}$. Then

$$\phi_{p,q}(\xi_*^{-1}(B, it))(v) = \tfrac{1}{2}\mathrm{i}tv + \phi_{p,q}(\lambda_*^{-1}(B))(v).$$

We conclude that

Proposition 3. *(M, g) admits a non trivial parallel spinor if and only if there exists $0 \neq v \in \Delta_{p,q}$ such that*

$$B \cdot v := \phi_{p,q}(\lambda_*^{-1}(B))(v) = -\tfrac{1}{2}\mathrm{i}tv, \quad \forall (B, it) \in \xi_*(\tilde{\mathcal{H}}) \subset \mathcal{H} \oplus \mathcal{H}_A, \tag{12}$$

where $\tilde{\mathcal{H}}$, $\mathcal{H}$ and $\mathcal{H}_A$ are respectively the Lie algebras of $\tilde{H}$, H and H_A.

3.2. Proof of theorem 3

Let $(M, g, S, P_{\mathbb{S}^1}, A)$ be a *spin*^{c} structure where (M, g) is a connected simply connected irreducible non-locally symmetric pseudo-Riemannian manifold of dimension $n = p + q$ and signature (p, q), which admits a non trivial parallel spinor ψ. We consider the two distributions T and E defined by

$$T_x := \{X \in T_xM; X \cdot \psi = 0\},$$

$$E_x = \{X \in T_xM; \exists Y \in T_xM; X \cdot \psi = \mathrm{i}Y \cdot \psi\},$$

for $x \in M$. Since ψ is parallel, By (8), T and E are parallel. Since T is isotropic and the manifold (M, g) is supposed irreducible, by the holonomy principe, we have

$$T = 0. \tag{13}$$

Now denote by F the image of the Ricci tensor:

$$F_x := \{Ric(X); X \in T_xM\}.$$

Since ψ is parallel, (11) shows that

$$Ric(X) \cdot \psi = \mathrm{i}\tilde{\omega}(X)\psi. \tag{14}$$

Then $F \subset E$. Consequently, from (13), we obtain

$$E^\perp \subset F^\perp = \{Y \in TM; Ric(Y) = 0\} = \{Y \in TM; \tilde{\omega}(Y) = 0\}.$$

(M, g) is supposed irreducible, by the holonomy principe, $E = 0$ or $E = TM$. If $E = 0$, then $F = 0$. This gives $Ric = 0$ and $\tilde{\omega} = 0$. According to Remark 1, (M, g) is spin and ψ is a parallel spinor on M. If $E = TM$, we have a (1,1)-tensor J defined by

$$X \cdot \psi = \mathrm{i}J(X) \cdot \psi, \quad \text{where} \quad X \in TM. \tag{15}$$

Lemma 1. *For $X, Y \in T_xM$, if $(X + \mathrm{i}Y) \cdot \psi = 0$ then $g(X, Y) = 0$ and $g(X, X) = g(Y, Y)$.*

Proof of Lemma 1. If we denote by g^c the complex form of g then

$$\begin{aligned}(X + \mathrm{i}Y) \cdot (X + \mathrm{i}Y) \cdot \psi &= -g^c(X + \mathrm{i}Y, X + \mathrm{i}Y)\psi \\ &= (-g(X, X) + g(Y, Y) - 2ig(X, Y))\psi = 0.\end{aligned}$$

Since ψ is non trivial we obtain the lemma.

Lemma 1 implies that J defines an orthogonal almost complex structure on M. Moreover, from (8) and (15) we obtain J is parallel, since ψ is parallel. In consequence, (M, g) is a Kähler manifold. Now if (M, g) is a Kähler manifold then there exists a canonical *spin*^{c} structure of (M, g). And from Remark 1 and Theorem 2, the following conditions are equivalent:

(a) (M, g) is not *spin*,
(b) $H_A = \mathbb{S}^1$,
(c) (M, g) is not Ricci-flat,
(d) $H = U(p', q')$.

Then the equivalence between (i) and (ii) of Theorem 3 are proved. And from Theorem 2, we have the equivalence between (ii) and (iii). To finish the proof of Theorem 3, it remains to show for $H = U(p', q')$ that the dimension of parallel spinors on M is 1. For this, we remark that if we reduce the principle bundles $P_{Spin^c(p,q)}$ and $P_{SO(p,q)} \tilde{\times} P_{\mathbb{S}^1}$ to their holonomy bundles the diagram (6) becomes

$$\begin{array}{ccc} & & \widetilde{H} \\ & \overset{\widetilde{\alpha}}{\nearrow} & \downarrow \xi \\ U(p',q') & \xrightarrow{\alpha} & \alpha(U(p',q')), \end{array} \tag{16}$$

and the holonomy group of $P_{SO(p,q)} \tilde{\times} P_{\mathbb{S}^1}$ is exactly $\alpha(U(p', q')) = \xi(\tilde{H})$. However $U(p', q') = SU(p', q') \times U_{\mathbb{S}^1}$, where

$$U_{\mathbb{S}^1} = \left\{ \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} ; \lambda \in \mathbb{S}^1 \right\},$$

$u(p', q') = su(p', q') \oplus u_{\mathbb{S}^1}$ where $u_{\mathbb{S}^1} \simeq \mathrm{i}\mathbb{R}$ is the Lie algebra of $U_{\mathbb{S}^1}$. If we consider the imbedding

$$\mathrm{i} : u(p', q') \hookrightarrow so(2p', 2q')$$

$$A + \mathrm{i}B = ((a_{kl})_{1\le k,l\le n} + \mathrm{i}(b_{kl})_{1\le k,l\le n}) \to \left(\begin{pmatrix} a_{kl} & b_{kl} \\ -b_{kl} & a_{kl} \end{pmatrix} \right)_{1\le k,l\le n}$$

$\alpha_*(u_{\mathbb{S}^1})$ is generated by (E_{12}, i). Then:

$$\xi_*(\tilde{\mathcal{H}}) = \alpha_*(u(p', q')) = \alpha_*(su(p', q')) \oplus \alpha_*(u_{\mathbb{S}^1}) = i(su(p', q')) \oplus (E_{12}, \mathrm{i})\mathbb{R}$$

From [1], $u^+ := u(1, \ldots, 1)$ and $u^- := u(-1, \ldots, -1)$ generate the space $V_{su(p',q')} = \{v \in \Delta_{p,q}; su(p', q')v = 0\}$. Moreover, by (1):

$$E_{12} \cdot u^+ = \tfrac{1}{2} e_1 \cdot e_2 \cdot u^+ = \tfrac{1}{2}\mathrm{i}u^+ \quad \text{and} \quad E_{12} \cdot u^- = -\tfrac{1}{2}\mathrm{i}u^-.$$

Then u^- belongs to the space

$$V_{u(p',q')} = \{v \in \Delta_{p,q}; B \cdot v + \tfrac{1}{2} itv = 0, \forall (B, it) \in \xi_*(\tilde{\mathcal{H}})\},$$

and it is easy to verify that u^- generates $V_{u(p',q')}$. This completes the proof of Theorem 3.

Remark 2. By Theorem 3, we deduce the results of Moroianau made in the riemannian case (see [13]).

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